Chapter 28: Sources of Magnetic Field

Sources are moving electric charges

single charged particle:

\[ \vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2} \]

\[ = k \frac{q\vec{v} \times \hat{r}}{r^2} \]

compare

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q\hat{r}}{r^2} \]

\[ = k \frac{q\hat{r}}{r^2} \]

field lines circulate around

“straight line trajectory”

from the definition of a Coulomb and other standards,

\[ \mu_0 = 4\pi \times 10^{-7} \text{ N s}^2/\text{C}^2 = 4\pi \times 10^{-7} \text{ T m/A} \]

with \( \varepsilon_0 \), the speed of light as a fundamental constant can be determined (in later chapters) from

\[ c^2 = \frac{1}{(\mu_0 \varepsilon_0)} \]
Consider: Two protons move in the x direction at a speed $v$.

Calculate *all* the forces each exerts on the other.

**Magnetic:**

$$F_{1 \text{ on } 2} = q_2 v_2 \times B$$

at due to 1

$$= q_2 v_2 \times (k' (q_1 v_1 \times r_{12})/(r_{12})^2)$$

**Electric:**

$$F_{1 \text{ on } 2} = (k q_1 q_2)/(r_{12})^2 r_{12}$$

Simple geometry (all 90 degree angles)

$$F_{\text{magnetic}} = k' |q_2 v_2 q_1 v_1| / (r_{12})^2$$

$$= k' e^2 v^2 / r^2$$

$$F_{\text{electric}} = k e^2 / r^2$$

$$F_{\text{magnetic}} / F_{\text{electric}} = k' v^2 / k = (\mu_0 / 4 \pi) v^2 / (1 / 4 \pi \varepsilon_0)$$

$$= v^2 / c^2$$

*but... depends upon frame?*

Gauss’s Law for magnetism:

Magnetic field lines encircle currents/moving charges. Field lines do not end or begin on any “charges”.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

for *any* closed surface.
Current elements as magnetic field sources
superposition of contributions of all charge carriers
+ large number of carriers + small current element

\[ \vec{B} = \sum \frac{\mu_o q_i \vec{v}_i \times \hat{r}_i}{4\pi r_i^2} \]
\[ = \frac{\mu_o n A d\ell}{4\pi} \frac{q \vec{v}_d \times \hat{r}}{r^2} \]
\[ = \frac{\mu_o}{4\pi} (n A d\ell) \frac{q \vec{v}_d \times \hat{r}}{r^2} \]

\[ d\vec{B} = \frac{\mu_o I d\ell \times \hat{r}}{4\pi r^2} \]
\[ \vec{B} = \frac{\mu_o}{4\pi} \int \frac{I d\ell \times \hat{r}}{r^2} \]

Field of a long straight wire

\[ d\vec{B} = \frac{\mu_o I d\ell \times \hat{r}}{4\pi r^2} \]
\[ = \frac{\mu_o I y \sin \theta}{4\pi r^2} (-\hat{z}) \text{ (into page)} \]
\[ = \frac{\mu_o I x}{4\pi \left(x^2 + y^2\right)^{3/2}} dy(-\hat{z}) \]
\[ \vec{B} = \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \frac{I x}{\left(x^2 + y^2\right)^{3/2}} dy(-\hat{z}) \]
\[ \vec{B} = \frac{\mu_o I}{2\pi} (-\hat{z}) \]
\[ B = \frac{\mu_o I}{2\pi r} \]
Example: A long straight wire carries a current of 100 A. At what distance will the magnetic field due to the wire be approximately as strong as the earth’s field ($10^{-4}$ T)?

Force between two long parallel current carrying wires (consider, for this example, currents in the same direction)

\[ F = I_2 B_{due \ to \ I_1} L \]

\[ = I_2 \left( \mu_0 \frac{I_1}{(2\pi r)} \right) L \]

\[ F/L = \mu_0 I_1 I_2 / (2\pi r) \]

force on $I_1$ is towards $I_2$

force is attractive (force is repulsive for currents in opposite directions!)

Example: Two 1m wires separated by 1cm each carry 10 A in the same direction. What is the force one wire exerts on the other
Magnetic Field of a Circular Current Loop

\[ |d\vec{B}| = k' \left( \frac{I}{r^2} \right) \left| \vec{d}\vec{l} \times \hat{r} \right| \]

\[ = \left( \frac{k' I dl}{r^2} \right) \quad \vec{d}\vec{l} \perp \hat{r} \]

\[ r = \sqrt{x^2 + a^2} \]

\[ \cos(90 - \theta) = \sin \theta \]

\[ = \frac{a}{\sqrt{x^2 + a^2}} \]

\[ dB_x = |d\vec{B}| \cos(90 - \theta) \]

\[ = \left( \frac{k' I dl}{x^2 + a^2} \sqrt{x^2 + a^2} \right) \frac{a}{\sqrt{x^2 + a^2}} \]

\[ dl = ds \text{ (segment of arc)} \]

\[ B_x = \int dB_x \]

\[ = \int k' \left( \frac{a dl}{(x^2 + a^2)^{3/2}} \right) = k' \frac{2\pi a I}{(x^2 + a^2)^{3/2}} \]

\[ B_x = k' \frac{2M}{(x^2 + a^2)^{3/2}} \]

\[ B_x = \frac{\mu_0 a^2 I}{2(x^2 + a^2)^{3/2}} \]

\[ B_x = \frac{\mu_0 I}{2a} \quad \text{at center} \]

\[ B_x \approx \frac{\mu_0 a^2 I}{2x^3} \quad x \gg a \]

\[ = k' \frac{2M}{x^3} \]
Example: A coil consisting of 100 circular loops .2 m in radius carries a current of 5 A. What is the magnetic field strength at the center?

N loops => N x magnetic field of 1 loop.

At what distance will the field strength be half that at the center?

**Ampere’s Law**

- Equivalent to Biot-Savart Law \( d\vec{B} = \frac{\mu_o}{4\pi} \frac{Idl \times \hat{r}}{r^2} \)
- Useful in areas of high symmetry
- Analogous to Gauss’s Law for Electric Fields

Formulated in terms of:

\( \oint \vec{B} \cdot d\vec{l} = \oint B_l dl \)

For simplicity, consider single long straight wire (source) and paths for the integral confined to a plane perpendicular to the wire.
\[ \vec{B} \cdot d\vec{l} = Bdl \cos \theta \]
\[ = Brd\theta \]
\[ = \frac{\mu_o I}{2\pi} \theta dl \]
\[ = \frac{\mu_o I}{2\pi} d\theta \]

\[ \oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_o I}{2\pi} d\theta \]
\[ = \mu_o I \text{ path encloses current} \]
\[ = 0 \text{ path does not enclose current} \]

\[ \oint \vec{B} \cdot d\vec{l} = \mu_o I_{\text{enclosed}} \]

Right Hand Rule relates conventions for directions of \( I \) and \( d\vec{l} \).

In general
\[ \oint \vec{B} \cdot d\vec{l} = \mu_o \int \vec{J} \cdot d\vec{A} \]

- Amperian Loop analogous to Gaussian surface
- Use paths with \( \mathbf{B} \) parallel/perpendicular to path
- Use paths which reflect symmetry
Application of Ampere’s Law: field of a long straight wire

Cylindrical Symmetry, field lines circulate around wire.

\[
\oint B(r) \cdot d\vec{l} = B(r) \oint dl
\]

\[
= B(r)2\pi r
\]

\[
= \mu_o I
\]

\[
B(r) = \frac{\mu_o I}{2\pi r}
\]

Field inside a long conductor

\[
\oint B(r) \cdot d\vec{l} = B(r) \oint dl
\]

\[
= B(r)2\pi r
\]

\[
= \mu_o I_{encl}
\]

\[
I_{encl} = JA
\]

\[
= \frac{I}{\pi R^2} \pi r^2
\]

\[
B(r)2\pi r = \frac{\mu_o I}{R^2} r^2
\]

\[
B(r) = \frac{\mu_o I}{2\pi R^2}
\]

\[
[B(r) = \frac{\mu_o I}{2\pi r} \text{ outside}]
\]
Homework: Coaxial cable
Magnetic Field in a Solenoid

Close packed stacks of coils form cylinder

Fields tend to cancel in region right between wires.
Field Lines continue down center of cylinder
Field is negligible directly outside of the cylinder

\[ B \propto nI \]

\[ \oint B \cdot d\vec{l} = BL \]

\[ I_{enc} = LnI \text{ (} n \text{ turns per unit length)} \]

\[ BL = \mu_o LnI \]

\[ B = \mu_o nI \]

\[ B \approx \mu_o \frac{N}{L} I \text{ (for finite solenoid)} \]
Example: what field is produced in an air core solenoid with 20 turns per cm carrying a current of 5A?

Toroidal Solenoid

Field along center of torus

\[ \oint \mathbf{B} \cdot d\mathbf{l} = B2\pi r \]

\[ I_{enc} = NI \]

\[ B2\pi r = \mu_o NI \]

\[ B = \frac{\mu_o NI}{2\pi r} \]

Charge trapped in a toroidal magnetic field

http://phys23p.sl.psu.edu/phys_anim/EM/charge_in_bfield_toroidal.mn.png
Magnetic Materials

Microscopic current loops:
- electron “orbits”
- electron “spin”

\[ I = e \frac{v}{2\pi r} \]

\[ \mu = IA = e \frac{v}{2\pi r} r^2 \]
\[ = e \frac{mv r}{2m} = e \frac{L}{2m} \]
\[ L = n \frac{\hbar}{2\pi} = n\hbar \]

\[ \mu_B = \frac{e}{2m} \hbar = \text{Bohr Magneton} \]
\[ = 9.274 \times 10^{-24} J/T \]
\[ U = -\mu \cdot \vec{B} \]
\[ \vec{\tau} = \vec{\mu} \times \vec{B} \]

Quantum Effects: quantized L, Pauli Exclusion Principle important in macroscopic magnetic behavior.

Magnetic Materials: Microscopic magnetic moments interact with an external (applied) magnetic field \( B_o \) and each other, producing additional contributions to the net magnetic field \( B \).

Magnetization \( M = \mu_{\text{tot}} / V \)

\[ B = B_o + \mu_o M \]

linear approximation: \( M \) proportional to \( B_o \)

\[ \mu_o \Rightarrow \mu = K_m \mu_o = \text{permeability} \]
\[ \chi_m = K_m -1 \text{ magnetic Susceptibility} \]

Types of Materials
- **Diamagnetic**: Magnetic field decreases in strength.
- **Paramagnetic**: Magnetic field increases in strength.
- **Ferromagnetic**: Magnetic field increases in strength!

Diamagnetic and Paramagnetic are often approximately linear with \( +/- \chi_m \)
Ferromagnetism:

Greatly increases field

Highly nonlinear, with Hysteresis:

Hysteresis = magnetic record

Magnetization forms in Magnetic Domains

Displacement Current

“Generalizing” displacement current for Ampere’s Law

Conduction current creates magnetic field

Amperian loop with surface

\[ I_{\text{enc}} = i_c \]

Parallel Plate Capacitor

Amperian loop with surface

\[ I_{\text{enc}} = 0??? \]

Parallel Plate Capacitor
Define Displacement Current between plates so that \( i_D = i_C \)

\[
q = Cv = \varepsilon \frac{A}{d} Ed = \varepsilon EA = \varepsilon \Phi_E
\]

\[
i_C = \frac{dq}{dt} = \varepsilon \frac{d\Phi_E}{dt} \equiv i_D
\]

\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 (i_C + i_D)
\]

\[
= \mu_0 i_C + \mu_0 \varepsilon \frac{d}{dt} \int \vec{E} \cdot d\vec{A}
\]

\[
\vec{j}_D = \varepsilon \frac{d\vec{E}}{dt}
\]

changing Electric Fields can create Magnetic Fields!