4: Two-Dimensional Kinematics

Kinematics in Two Dimensions

Rates of change:

- **Average velocity**
  \[
  \bar{v} = \frac{\Delta \mathbf{r}}{\Delta t}
  \]

Change in vector \(\Rightarrow\) change in components!

- **Instantaneous velocity and acceleration**
  \[
  \mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}
  \]
  \[
  \mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}
  \]

Vectors!

\[
\mathbf{v} = \sqrt{v_x^2 + v_y^2}
\]
\[
\mathbf{a} = \sqrt{a_x^2 + a_y^2}
\]

2-D kinematics:

\[
\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t
\]
\[
\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2
\]

- \(v_x = v_{0x} + a_x t\)
- \(v_y = v_{0y} + a_y t\)
- \(x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2\)
- \(y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2\)

but look at components...
The “Fab Four” revisited

\[ \begin{align*}
\text{\textit{x-direction}} & \\
\Delta x &= \frac{v_x + v_{x0}}{2} t \\
\Delta x &= v_{x0} t + \frac{1}{2} a_x t^2 \\
v_x &= v_{x0} + a_x t \\
v_x^2 &= v_{x0}^2 + 2 a_x \Delta x \\
\text{\textit{y-direction}} & \\
\Delta y &= \frac{v_y + v_{y0}}{2} t \\
\Delta y &= v_{y0} t + \frac{1}{2} a_y t^2 \\
v_y &= v_{y0} + a_y t \\
v_y^2 &= v_{y0}^2 + 2 a_y \Delta y
\end{align*} \]

the \( x \) and \( y \) directions are connected by time \( t \)

Example: A hummingbird is flying such that it is initially moving vertically upwards with a speed of 4.6 m/s and is accelerating horizontally at 11 m/s\(^2\). What is the bird's displacement after 0.55 s?
Projectile Motion:

- take acceleration to be due solely to gravity
- by convention, take x horizontal and y vertical (positive upwards)

\[ \begin{align*}
x - \text{direction} & \\
\Delta x &= v_{x0} t \\
v_x &= v_{x0}
\end{align*} \]

\[ \begin{align*}
y - \text{direction} & \\
y &= y_0 + \frac{v_y + v_{y0}}{2} t \\
y &= y_0 + v_{y0} t - \frac{1}{2} g t^2 \\
v_y &= v_{y0} - g t \\
v_y^2 &= v_{y0}^2 - 2 g \Delta y
\end{align*} \]

initial velocity form speed and launch angle (from horizontal)

\[ \begin{align*}
v_{x0} &= v_0 \cos \theta \\
v_{y0} &= v_0 \sin \theta
\end{align*} \]

for a horizontal launch \((\theta = 0)\) from an initial height \(h\)

\[ \begin{align*}
x &= v_o t \\
\Delta x &= v_o = \text{constant} \\
v_x &= v_0 = \text{constant} \\
v_y &= -g t \\
v_y^2 &= -2 g \Delta y
\end{align*} \]
Example: (not from book, similar to example 4.4) A ball rolls off of a horizontal table 1.20 m high, and lands 1.80 m beyond the edge of the table. How long is the ball in the air? What was the ball's initial velocity?
Projectile Motion, general angle:

take launch point as origin

$x$-direction $\quad y$-direction

\[ x = v_{x0} t \quad y = v_{y0} t - \frac{1}{2} g t^2 \]
\[ v_x = v_{x0} \quad v_y = v_{y0} - g t \]
\[ v_y^2 = v_{y0}^2 - 2 g y \]

eliminate $t$ from $y$ equation

\[ t = \frac{x}{v_{x0}} = \frac{x}{v_0 \cos \theta} \]
\[ y = v_{y0} t - \frac{1}{2} g t^2 \]
\[ = v_0 \sin \theta t - \frac{1}{2} g t^2 \]
\[ = v_0 \sin \theta \frac{x}{v_0 \cos \theta} - \frac{1}{2} g \left( \frac{x}{v_0 \cos \theta} \right)^2 \]
\[ = v_0 \tan \theta x - \frac{g}{2 (v_0 \cos \theta)^2} x^2 \]

trajectory shape is parabola!
Example: A projectile is launched from the origin with an initial speed of 20.0 m/s and an angle of 35° above the horizontal. Find its position at 0.500 s, 1.00 s, and 1.50 s. Find its velocity at 0.500 s, 1.00 s, and 1.50 s.

A golfer sends a ball over a 3.00 m tree that is 14.0 m away. The ball lands on the green (same level as original shot) after traveling a horizontal distance of 17.8 m. If the ball leaves the club at an angle of 54.0° above the horizon and lands on the green 2.24 s, what was the initial speed of the ball? How high was the ball when it passed over the tree?
Example: A golfer hits the ball with an initial speed of 30.0 m/s at an angle of 50.0°. The ball lands on the green which is 5.00 m above the level where the ball was struck. How long is the ball in the air? How far has the ball traveled horizontally? What is the speed and direction of motion of the ball when it lands?
Projectile Range

![Symmetric Trajectory]

\[ y = v_{0y} t - \frac{1}{2} gt^2 = 0 \] at impact \( \Rightarrow t = \frac{2v_{0y}}{g} \)

\[ R = x = v_{0x} t = v_{0x} \frac{2v_{0y}}{g} \]

\[ = v_0 \cos\theta \frac{2v_0 \sin\theta}{g} = 2 \frac{v_0^2 \sin\theta \cos\theta}{g} \]

\[ R = \frac{v_0^2 \sin 2\theta}{g} \]

\( \theta_2 = 90 - \theta_1 \) (angles with the same range)

\[ R_{\text{max}} = \frac{v_0^2}{g} \text{ (at } 45^\circ) \]

\[ y_{\text{max}} = \text{height at halfway point}(t / 2) = v_{0y} t - \frac{1}{2} gt^2 = 0 \] at \( \Rightarrow t = \frac{v_{0y}}{g} \)

\[ y_{\text{max}} = \frac{v_0^2 \sin^2\theta}{2g} \]
Example: A kickoff at a football game travels a horizontal distance of 45 yd. If the ball was kicked at an angle of 40.0°, what was its initial speed?