Notes: Following a parameterized path

These are some quick notes on following a parameterized curve with an application for displaying incompressible fluid flow. Start with a path parameterized by a set of coordinate functions:

$$x = f_x(s)$$
$$y = f_y(s)$$
$$z = f_z(s)$$
$$\vec{r}(s) = f_x(s)\hat{i} + f_y(s)\hat{j} + f_z(s)\hat{k}$$

For motion, the speed along the path is parameterized by $v(s)$ and for fluid flow, the radius of the fluid is parameterized by R(s). Things would be simpler if the parameterization of the path preserved distances, i.e.

$$\left|\frac{d\vec{r}(s)}{ds}\right| = constant$$

but that is overly restrictive for simply constructed paths in general.

Motion:

Consider the motion of an object along the path, at a time *t* for an interval *dt*. In that small time interval *dt*, the object will also "move" along the parameterizations *s* by an amount *ds*. The object's instantaneous velocity is given by

$$\vec{v}(s) = \frac{d\vec{r}(s)}{ds}\frac{v(s)}{\left|\frac{d\vec{r}(s)}{ds}\right|} \quad .$$

In an interval *dt* the physical displacement will be

$$d\vec{r} = \vec{v}(s)\,dt$$

and for the corresponding parameter interval *ds*:

$$d\vec{r} = \frac{d\vec{r}(s)}{ds}\,ds$$

which gives the following relations between *ds* and *dt*:

$$ds = \frac{v(s)}{\left|\frac{d\vec{r}(s)}{ds}\right|}\,dt$$

In terms of mapping textures, map the segment from the uniformly textured cylinder of length *du* to the physical segment $dr = v(s)\,dt$, then translate the texture along the cylinder to mimic flow. The speed $v(s)$ becomes a relative speed of the fluid.

Field Lines

Field lines follow a path in space such that the field at any particular point in space is tangent to the path. If we parameterize the path as $\vec{u}(s)$ that corresponds to a field line of the path $\vec{E}(\vec{r})$ then the "constant velocity" traversal of that path is determined by

$$d\vec{u}=\frac{\vec{E}(\vec{u}(s))}{|\vec{E}(\vec{u}(s))|}ds$$